ELEMENTARY PROPERTIES OF THE t_ν-FUNCTIONS*

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The uniformization of the elliptic integral of the first species is given in classical form by the following expression of Weierstrass:

$$\lim_{n\to\infty} \sum_{h,k}^{h^2+k^2<n} (u + \omega_1 h + \omega_2 k)^{-2} = \wp\left(u, \frac{\omega_1}{2}, \frac{\omega_2}{2}\right).$$

This leads to building up rationally an initial value of this $\wp$-function from the coefficients of the given biquadratic polynomial. The projective invariants of biquadratic forms and their covariants are connected through identical relations which may be regarded as characteristic functional equations for elliptic functions. However, this method does not seem to reduce matters to their elements.

Making use of some earlier investigations‡ we shall link up the above uniformization with certain double series of the following form:

$$\lim_{n\to\infty} \sum_{h,k}^{0<h^2+k^2<n} \frac{e^{2\pi i(xh+yk)}}{(h\omega_1 + k\omega_2)^\nu} = t_\nu(x, y) \qquad (\nu = 1, 2, \cdots).$$

By means of this expression we are able to formulate the known properties of elliptic functions in a more natural way. In particular we shall derive the algebraic relation between $t_1(x, y)$ and $t_1(x/2, y/2)$ which is the simplest case of a division theory, the latter containing the corresponding developments of Abel.

Moreover, these functions seem to be of interest for their own sake because of the existence of relations of the following type:

$$t_4\left\{x + \frac{1}{\pi i} t_1\left(\frac{x}{2}, \frac{y}{2}\right), \; y + \frac{\omega}{\pi i} t_1\left(\frac{x}{2}, \frac{y}{2}\right)\right\} = 0,$$

which for $\omega = \rho^{2\pi i/3}$ holds identically in x, y.

Let w be complex, $\nu = 0, 1, 2, 3, 4$, and τ_ν complex. We form the polynomial biquadratic in w

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$$\sum_{\nu=0}^{4} \frac{w^\nu}{\nu!}\tau_{4-\nu} = V(w), \tag{1}$$

and the expressions

$$24\tau_0\tau_4 - 24\tau_1\tau_3 + 12\tau_2^2 = (24)^2 \cdot S, \tag{2}$$

$$48\tau_0\tau_2\tau_4 + 24\tau_1\tau_2\tau_3 - 8\tau_2^2 - 36\tau_0\tau_3^2 - 24\tau_1^2\tau_4 = (24)^3 \cdot T, \tag{2'}$$

which are invariant under linear fractional transformations of w.

Now the function $V^{1/2}$ which is two-valued in w is to be made one-valued in its dependence on a complex variable u by means of elliptic functions. The latter can be introduced in connection with two complex auxiliary magnitudes ω_1, ω_2 restricted by $F(\omega_2/\omega_1) > 0$. Using the notation

$$\sum_{h,k}^{0<h^2+k^2} = \sum_{h,k}{}'$$

we write a system of two transcendental determining equations

$$\begin{aligned} \left(\frac{24}{\tau_0}\right)^2 \frac{S}{60} &= \sum{}'(\omega_1 h + \omega_2 k)^{-4}, \\ \left(\frac{24}{\tau_0}\right)^3 \frac{T}{140} &= \sum{}'(\omega_1 h + \omega_2 k)^{-6}, \end{aligned} \tag{3}$$

and we construct in the customary way the Weierstrassian expression

$$\lim_{n\to\infty} \sum_{h,k}^{h^2+k^2<n} (u + \omega_1 h + \omega_2 k)^{-2} = \wp\left(u, \frac{\omega_1}{2}, \frac{\omega_2}{2}\right) = \wp(u).$$

The complex number z is then found as the unique solution mod ω_1, ω_2 of the additional determining equation

$$\tau_1^2 - 2\tau_0\tau_2 = \tau_0^2 \cdot \wp(z), \tag{4}$$

and it also appears that

$$\tau_0^3\,\wp'(z) = 2\tau_1^3 - 6\tau_0\tau_1\tau_2 + 6\tau_1^2\tau_3. \tag{4'}$$

The above uniformization is then expressed by

$$w = -\frac{\tau_1}{\tau_0} + \frac{1}{2}\,\frac{\wp'(u) - \wp'(z)}{\wp(u) - \wp(z)}, \tag{5}$$

$$V^{1/2} = \tau_0^{1/2}[\wp(u) - \wp(u + z)]. \tag{5'}$$

Between the two statements (5), (5′) there exists moreover the connection

$$\frac{1}{2}\frac{d}{du}\frac{\wp'(u)-\wp'(z)}{\wp(u)-\wp(z)}=\wp(u)-\wp(u+z), \tag{6}$$

which, in consequence of the addition theorem, leads to the integral representation

$$\tau_0^{1/2}\int\frac{dw}{V^{1/2}}=u(w)=u.$$

This whole situation suggests a further problem concerning the number of the undetermined quantities involved. Without loss in generality we may place $\tau_0=-1$; then through a unimodular linear transformation the quadruple of coefficients $\tau_1, \cdots, \tau_4$ in (1) is simultaneously transformed into another system $\tau_1^*, \cdots, \tau_4^*$; in this way the quantities on the right in (2), (2′) remain unchanged. Now in making, according to the classical theory, the transition from the old to the new quadruple we have two degrees of freedom. These we wish to put in evidence. We succeed in doing so by introducing four new parameters x, y; Ω_1, Ω_2 through the following conditions:

(I) There is a one-to-one correspondence between $\tau_1, \cdots, \tau_4$ and x, y; Ω_1, Ω_2.

$$\frac{\partial S}{\partial x}=\frac{\partial S}{\partial y}=\frac{\partial T}{\partial x}=\frac{\partial T}{\partial y}=0. \tag{II}$$

In other words, we wish to map the τ-quadruple on two number pairs in such a way that the statements (2), (2′) hold identically in one of them. This plan we carry out by making use of the periods implied by (3) placing $\Omega_j=\omega_j$ for $j=1, 2$. Furthermore let $0<x<1$, $0<y<1$;

$$\lim_{n\to\infty}\sum_{h,k}^{0<h^2+k^2<n}=\sum{}' \qquad (\nu=1,2,3,\cdots);$$

$$\sum{}'\frac{e^{2\pi i(xh+yk)}}{(\omega_1h+\omega_2k)^\nu}=t_\nu(x,y)=t_\nu. \tag{7}$$

In particular for $\nu=4$ there exists

$$\lim_{x,y\to 0} t_4=t_4(0,0). \tag{8}$$

Also, in addition to the previous condition $\tau_0=-1$ we place $\tau_\lambda=t_\lambda$ for $\lambda=1$, 2, 3 and $\tau_4=t_4-t_4(0,0)$. As analogous to (7), we still use the modular form $t_6(0,0)$, which is of the -6th dimension. The functions t_ν introduced by (7) occupy an intermediate position between modular forms and elliptic functions. The latter can be represented rationally through the t_ν, but not conversely. The expressions (7) lead to a theory which goes beyond the results of Weierstrass. They represent meromorphic functions in x and y which are successively joined by the differential relation

$$\left\{\frac{\omega_1}{2\pi i}\frac{\partial}{\partial x}+\frac{\omega_2}{2\pi i}\frac{\partial}{\partial y}\right\}t_\nu = t_{\nu-1}, \tag{9}$$

which for a given $t_{\nu-1}$ admits of the determination of t_ν but for an additive term. The latter is not only meromorphic in each of the variables x, y but is even an analytic function of the single argument $(\omega_1 y-\omega_2 x)$. If we add as a condition in the large (Bedingung im Grossen) the periodicity

$$t_\nu(x, y) = t_\nu(x+1, y) = t_\nu(x, y+1), \tag{10}$$

then this meromorphic term is found to be twofold periodic, that is, elliptic, and hence to be characterized through limiting conditions. The initial value (8), together with relation (9), the periods (10), and e.g. the limiting condition

$$\lim_{x,y\to 0}\left\{\frac{1}{\omega_1 y-\omega_2 x} - t_1\right\} = 0, \tag{10'}$$

is sufficient to determine $t_1(x, y)$.

The above t_ν are not mutually independent but are connected by algebraic recursions. Among others we have

$$\begin{aligned}9t_3^2 + 6t_3[t_1^3 + 3t_1t_2] - \{8t_2^3 + 3t_1^2t_2^2 \\ - 15t_4(0, 0)[t_1^2 + 2t_2] + 35t_6(0, 0)\} = 0,\end{aligned} \tag{11}$$

and

$$t_2^2 - 2t_1t_3 - 2t_4 - 3t_4(0, 0) = 0, \tag{11'}$$

which allow us to transform the invariants (2),(2′) of the polynomial (1). According to (7), (8) the individual summands on the left side of (2),(2′) depend on x, y; their sums, however, as seen from (3), are independent of x, y, a fact which we recognize as equivalent to (11), (11′). For

$$-24[t_4 - t_4(0,0)] - 24t_1t_3 + 12t_2^2 = 12[t_2^2 - 2t_1t_3 - 2t_4 + 2t_4(0,0)] = 60t_4(0,0),$$

$$\begin{aligned}&- 48t_2[t_4 - t_4(0, 0)] + 24t_1t_2t_3 - 8t_2^3 + 36t_3^2 + 24t_1^2[t_4(0, 0) - t_4] \\ &= 12[t_1^2 + 2t_2]\{5t_4(0, 0) + 2t_1t_3 - t_2^2\} + 24t_1t_2t_3 - 8t_2^3 + 36t_3^2 \\ &= - 140t_6(0, 0).\end{aligned}$$

Moreover from (4), (4′), (7), (8) we infer

$$\begin{aligned}t_1^2 + 2t_2 &= \wp(z), \\ 2t_1^3 + 6t_1t_2 + 6t_3 + \wp'(z) &= 0,\end{aligned} \tag{12}$$

and from (10), (10′) obtain the value of the argument in (4),

$$z = \omega_1 y - \omega_2 x. \tag{13}$$

Substituting this value into the following identities:

$$t_1 + \frac{1}{2}\,\frac{\wp'(u) - \wp'(z)}{\wp(u) - \wp(z)} = w; \quad \wp(u) - \wp(u+z) = -V^{1/2}; \tag{14}$$

$$\sum_{\nu=0}^{4} \frac{w^\nu}{\nu!} t_{4-\nu} = t_4(0, 0) + V(w), \tag{15}$$

we find the addition theorem of the $\wp$-function translated into the t_ν-language.

If the τ_ν quadruple is given through (1), the invariants may be computed from (11),(11′); and $\wp(z)$ from (12); and thus a suitable triplet ω_1, ω_2, z can be determined as usual. The real values x, y then follow uniquely from (13).

If $\lambda = 1, 2, \cdots$, for a sufficiently small $|w|$ and for $\omega_1 y - \omega_2 x \not\equiv 0 \pmod{\omega_1, \omega_2}$ Taylor's theorem in two variables furnishes, as an immediate consequence of the differential relation of the original series,

$$\begin{aligned} t_\lambda\left(x + w\frac{\omega_1}{2\pi i},\ y + w\frac{\omega_2}{2\pi i}\right) &= \sum_{\nu=0}^{\infty} \frac{w^\nu}{\nu!}\left\{\frac{\omega_1}{2\pi i}\frac{\partial}{\partial x} + \frac{\omega_2}{2\pi i}\frac{\partial}{\partial y}\right\}^\nu t_\lambda(x, y) \\ &= \sum_{\nu=0}^{\infty} \frac{w^\nu}{\nu!} t_{\lambda-\nu}(x, y) \\ &= \sum_{\nu=0}^{\lambda} \frac{w^\nu}{\nu!} t_{\lambda-\nu}(x, y). \end{aligned} \tag{16}$$

On the other hand if we develop a power series with indeterminate coefficients we obtain algebraic recursions between any three t_λ so that the identity (16) yields an independent theory of the t_λ functions.

The special case $\lambda = 4$ gives

$$V(w) = t_4\left(x + w\frac{\omega_1}{2\pi i},\ y + w\frac{\omega_2}{2\pi i}\right) - t_4(0, 0). \tag{17}$$

Studying the analytic continuation* of this function into the two complex y and x-planes we find that the right side of (17) is a function of z only, and even elliptic in z. By putting

$$u = \omega_1 r - \omega_2 q \tag{13′}$$

* See Mathematische Zeitschrift, loc. cit.

which means a splitting of u according to periods, we can give a linear representation

$$w = t_1(q + x, r + y) - t_1(q, r) \tag{14'}$$

where w is itself an argument of the similar function (17).

We next wish to emphasize the fact which characterizes the elliptic case, that two projective independent invariants of V remain invariant also under continuous x, y transformations. For this purpose we introduce further the covariants of V and the algebraic relation connecting them. In forming successive derivatives with respect to w, we are lead to the Hessian function which, as V, is elliptic both in u and z. Moreover this covariant H has two further properties in common with the original polynomial: The operation (9) transforms both times each coefficient into its right neighbor; to the final coefficient $\tau_0 = -1$ in (1) corresponds $2t_1^2 + 4t_2$ in (16′). Starting from the two simplest elliptic functions 1 and $\wp'$ we thus reach, after four integrations with suitable determination of the constants, the functions V and H respectively. A similar situation prevails for the Jacobian; the degree of its coefficients is increased by 3 in t_1, the order of the form by 2 in w.

The determination of these constants we shall not carry out here; but we nevertheless note that, taking the first statement in (16′), we get

$$H = 3t_3^2\left(x + w\frac{\omega_1}{2\pi i},\ y + w\frac{\omega_2}{2\pi i}\right)$$
$$+ 4\left[t_4(0, 0) - t_4\left(x + w\frac{\omega_1}{2\pi i},\ y + w\frac{\omega_2}{2\pi i}\right)\right]t_2\left(x + w\frac{\omega_1}{2\pi i},\ y + w\frac{\omega_2}{2\pi i}\right)$$

where the t_2, t_4 may be eliminated by means of the recursion

$$5[t_6 - t_6(0, 0)] + 3t_2[t_4 - t_4(0, 0)] = 2t_3^2.$$

Now let us consider the biquadratic equation

$$V(w) = 0, \tag{17'}$$

which determines discrete values of w. The corresponding determining equation for u is, according to (14),

$$\wp(u) = \wp(u + z).$$

The arguments in this equation are connected by the relations

$$-u \equiv u + z, \quad u \equiv -\frac{z}{2} \qquad (\mathrm{mod}\ \omega_1, \omega_2).$$

On the other hand (17′) will be satisfied, according to (10), (14′), by

$$w = t_1\left(\frac{x}{2}, \frac{y}{2}\right) - t_1\left(-\frac{x}{2}, -\frac{y}{2}\right) = 2t_1\left(\frac{x}{2}, \frac{y}{2},\right) \tag{18}$$

and the pairs of arguments

$$\left(\frac{x+1}{2}, \frac{y}{2}\right); \quad \left(\frac{x}{2}, \frac{y+1}{2}\right); \quad \left(\frac{x+1}{2}, \frac{y+1}{2}\right)$$

together with (18) furnish the complete system of solutions to (17′).

This is the simplest example of the division theory of the functions t_ν. It corresponds to Abel's theory of the division of arguments for elliptic functions. As in the case of the t_1 function the bisection is accomplished through an equation of the fourth degree; but in the present theory the structure of the coefficients used is more obvious.

The limiting case $|\omega_2| \to \infty$ admits of a simple check on (17′), (18); the expression (7) becomes a simply infinite series whose sum is a polynomial in x. In agreement with (1), (17′), (18) we have in particular,

$$\begin{aligned} \omega_1 w = 2\pi i(1 - x); \quad \omega_1\tau_1 &= \pi i(1 - 2x), \\ \omega_1^2\tau_2 &= \frac{\pi^2}{3}(1 - 6x + 6x^2), \\ \omega_1^3\tau_3 &= \frac{2\pi^3 i}{3}(x - 3x^2 + 2x^3), \\ \omega_1^4\tau_4 &= -\frac{2\pi^4}{3}(x^2 - 2x^3 + x^4). \end{aligned}$$

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